CASTELNUOVO-MUMFORD REGULARITY OF ASSOCIATED GRADED MODULES IN DIMENSION ONE

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ABSTRACT. An upper bound for the Castelnuovo-Mumford regularity of the associated graded module of an one-dimension module is given in term of its Hilbert coefficients. It is also investigated when the bound is attained.

INTRODUCTION

Let A be a local ring with the maximal ideal $\mathfrak{m}$ and M be a finitely generated A-module of dimension r. Denote by $G_I(M) = \oplus_{n \geq 0} I^n M / I^{n+1} M$ the associated graded module of M with respect to an $\mathfrak{m}$-primary ideal I. Because of the importance of the Castelnuovo-Mumford regularity $\mathrm{reg}(G_I(M))$ of $G_I(M)$, it is of interest to bound it in terms of other invariants of M. Rossi, Trung and Valla [12] and Linh [11] gave bounds on $\mathrm{reg}(G_I(M))$ in terms of the so-called extended degree of M with respect to I.

On the other hand, Trivedi [14] and Brodmann and Sharp [1] gave bounds for so-called Castelnuovo-Mumford regularity at and above level 1 in terms of the Hilbert coefficients $e_0(I, M), e_1(I, M), ..., e_{r-1}(I, M)$. Using also $e_r(I, M)$, Dung and Hoa [5] can bound $\mathrm{reg}(G_I(M))$. Bounds in [1, 5, 14] are very big: they are exponential functions of $r!$. In the general case, it follows from [9, Lemma 11 and Proposition 12] that in the worst case $\mathrm{reg}(G_I(M))$ must be a double exponential function of r. However one may hope that in small dimensions, one can give small and sharp bounds.

The aim of this paper is to give a sharp bound in dimension one case.

Theorem *Let M be an one-dimensional module. Let b be the maximal integer such that $IM \subseteq \mathfrak{m}^b M$. Then*

$$\mathrm{reg}(G_I(M)) \leq \binom{e_0 - b + 2}{2} - e_1 - 1.$$

We also give characterizations for the case that the bound is attained (see Theorem 2.3 and Theorem 2.5).

The paper is divided into two sections. In Section 1 we start with a few preliminary results on bounding the Castelnuovo-Mumford regularity of graded modules of dimension at most one. Then we apply these results to derive a bound on $\text{reg}(G_I(M))$ provided M is a Cohen-Macaulay module of dimension one (Proposition 1.9). Using a relation between Hilbert coefficients $e_0(I, M)$ and $e_1(I, M)$ given in Rossi-Valla [13] we can deduce and prove the above bound.

In Section 2 we give characterizations for the case that the bound is attained. The characterization is given in terms of the Hilbert-Poincare series (Theorem 2.5). In the case of Cohen-Macaulay modules there is also a characterization in terms of the Hilbert coefficients (Theorem 2.3).

1. AN UPPER BOUND

Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian standard graded ring over a local Artinian ring $(R_0, \mathfrak{m}_0)$. Since tensoring with $(R_0/\mathfrak{m}_0)(x)$ doesn't change invariants considered in this paper, without loss of generality we may always assume that $R_0/\mathfrak{m}_0$ is an infinite field. Let E be a finitely generated graded module of dimension r.

First let us recall some notation. For $0 \leq i \leq r$, put

$$a_i(E) = \sup\{n| \ H^i_{R_+}(E)_n \neq 0\},$$

where $R_+ = \oplus_{n>0} R_n$. The *Castelnuovo-Mumford regularity* of E is defined by

$$\text{reg}(E) = \max\{a_i(E) + i \mid 0 \leq i \leq r\},$$

and the *Castelnuovo-Mumford regularity of E at and above level* 1, is defined by

$$\text{reg}^1(E) = \max\{a_i(E) + i \mid 1 \leq i \leq r\}.$$

We denote the Hilbert function $\ell_{R_0}(E_t)$ and the Hilbert polynomial of E by $h_E(t)$ and $p_E(t)$, respectively. Writing $p_E(t)$ in the form:

$$p_E(t) = \sum_{i=0}^{r-1} (-1)^i e_i(E) \binom{t+r-1-i}{r-1-i},$$

we call the numbers $e_i(E)$ *Hilbert coefficients* of E.

We know that $h_E(t) = p_E(t)$ for all $t \gg 0$. The *postulation number* of a finitely generated graded R-module E is defined as the number

$$p(E) := \max\{t \mid h_E(t) \neq p_E(t)\}.$$

This number can be read off from the *Hilbert-Poincare series* of E, which is defined as follows:

$$HP_E(z) = \sum_{i \geq 0} h_E(i) z^i.$$

The following result is well-known, see e. g. [2, Lemma 4.1.7 and Proposition 4.1.12].

Lemma 1.1. *There is a polynomial $Q_E(z) \in \mathbb{Z}[z]$ such that $Q_E(1) \neq 0$ and*

$$HP_E(z) = \frac{Q_E(z)}{(1-z)^r}.$$

Moreover, $p(E) = \deg(Q_E(z)) - r$.

Remark 1.2. From the Grothendieck-Serre formula

$$(1) \qquad h_E(t) - p_E(t) = \sum_{i=0}^{r} (-1)^i \ell(H^i_{R_+}(E)_t),$$

we easily get

    (i) If $\operatorname{depth}(E) = 0$ then $p(E) \le \operatorname{reg}(E)$;
   (ii) If $\operatorname{depth}(E) > 0$ then $p(E) \le \operatorname{reg}(E) - 1$.

  Let $d(E)$ denote the maximal generating degree of E. For short, we also write $d := d(E)$ and $e := e_0(E)$. The proof of the following result is similar to that of [8, Lemma 4.2].

Lemma 1.3. *Assume that* $\dim(E) = 0$. *Let* $q := \sum_{i \le d} \ell(E_i)$. *Then*

    (i) $\operatorname{reg}(E) \le d + e - q$.
   (ii) *The following conditions are equivalent:*
      (a) $\operatorname{reg}(E) = d + e - q$;
      (b) $h_E(t) = \begin{cases} 1 & \text{if } d+1 \le t \le d+e-q, \\ 0 & \text{if } t \ge d+e-q+1. \end{cases}$

Proof. (i) Let $m = \operatorname{reg}(E)$. Since $\dim(E) = 0$, $m = \max\{t \mid E_t \ne 0\}$. Note that $E_i \ne 0$ for all $d \le i \le m$. Hence

$$(2) \qquad m \le d + \ell(E_{d+1} \oplus \cdots \oplus E_m) = d + \ell(E) - q = d + e - q.$$

(ii) By (2) is it clear that $m = d + e - q$ if only if $\ell(E_i) = 1$ for all $d+1 \le i \le m$. Since $\ell(E_i) = 0$ for all $i > m$, we get the equivalence of (a) and (b). $\qquad\square$

Lemma 1.4. *Let* E *be an one-dimensional Cohen-Macalay module. Let* $z \in R_1$ *be an* E-regular element and $\rho := \ell(E_{d(E/zE)})$. *Then*

    (i) $\operatorname{reg}(E) \le d + e - \rho$.
   (ii) *The following conditions are equivalent:*
      (a) $\operatorname{reg}(E) = d + e - \rho$;
      (b) $h_E(t) = \begin{cases} t - d + \rho & \text{if } d+1 \le t \le d+e-\rho-1, \\ e & \text{if } t \ge d+e-\rho; \end{cases}$
      (c) $p(E) = d + e - \rho - 1$.

Proof. (i) Since $z \in R_1$ is an E-regular element, we have $e(E/zE) = e$, $\operatorname{reg}(E/zE) = \operatorname{reg}(E)$ and

$$(3) \qquad \sum_{i \le t} h_{E/zE}(i) = h_E(t).$$

In particular $\sum_{i \le d(E/zE)} h_{E/zE}(i) = \rho$. Note that $d(E/zE) \le d(E)$. By Lemma 1.3,

$$(4) \qquad \operatorname{reg}(E/zE) \le d(E/zE) + e(E/zE) - \sum_{i \le d(E/zE)} h_{E/zE}(i) \le d + e - \rho.$$

Hence $\operatorname{reg}(E) \le d + e - \rho$.

  (ii) (a) $\Longrightarrow$ (b): If $\operatorname{reg}(E) = d+e-\rho$, then from (4) it implies that $d(E/zE) = d(E)$ and $\operatorname{reg}(E/zE) = d(E/zE) + e - \rho$. Using Lemma 1.3 (ii) and (3) we get (b).
   (b) $\Longrightarrow$ (c) follows from Lemma 1.1.

(c) $\implies$ (a): Suppose that $p(E) = d + e - \rho - 1$. By Remark 1.2 (ii), we get $d + e - \rho \leq \text{reg}(E)$. Since $\text{reg}(E) \leq d + e - \rho$, we get $\text{reg}(E) = d + e - \rho$. $\qquad\square$

Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field $K := A/\mathfrak{m}$ and M a finitely generated A-module of dimension r. In this paper, we always assume that I is an $\mathfrak{m}$-primary ideal. The *associated graded module* of M with respect to I is defined by

$$G_I(M) = \bigoplus_{n \geq 0} I^n M / I^{n+1} M.$$

This is a module over the associated graded ring $G = \bigoplus_{n \geq 0} I^n / I^{n+1}$. Let $HP_{I,M}(z) := HP_{G_I(M)}(z)$. We call $H_{I,M}(n) = \ell(M/I^{n+1}M)$ the Hilbert-Samuel function of M w.r.t. I. This function agrees with a polynomial - called Hilbert-Samuel polynomial and denote by $P_{I,M}(n)$ - for $n \gg 0$. If we write

$$P_{I,M}(n) = e_0(I, M)\binom{n + r}{r} - e_1(I, M)\binom{n + r - 1}{r - 1} + \cdots + (-1)^r e_r(I, M),$$

then the integers $e_i := e_i(I, M)$ for all $i = 1, ..., r$ are called *Hilbert coefficients* of M with respect to I (see [13, Section 1]).

Assume that M is a Cohen-Macaulay module. Kirby and Mehran [10] were able to show that $e_1 \leq \binom{e_0}{2}$. This result was improved by Rossi-Valla as follows:

Lemma 1.5. [13, Proposition 2.8] *Let M be an one-dimensional Cohen-Macalay module. Let b be a positive integer such that $IM \subseteq \mathfrak{m}^b M$. Then*

(i) $e_1 \leq \binom{e_0 - b + 1}{2}$.
(ii) *The following conditions are equivalent:*
   (a) $e_1 = \binom{e_0 - b + 1}{2}$;
   (b) $HP_{I,M}(z) = \frac{b + \sum_{i=1}^{e_0 - b} z^i}{1 - z}$.

Note that (ii) is not formulated in [13, Proposition 2.8], but it immediately follows from the proof of that result.

Remark 1.6. (i) If $e_1 = \binom{e_0 - b + 1}{2}$, then from (i) of the above lemma it follows that $b = \max\{t \mid IM \subseteq \mathfrak{m}^t M\}$.

(ii) If $I = \mathfrak{m}$ and $M = A$ then $b = 1$ and Lemma 1.5 (ii) is [7, Theorem 3.1].

Set $\overline{G_I(M)} := G_I(M)/H^0_{G_+}(G_I(M))$.

Lemma 1.7. *Let M be an one-dimensional module. Let b be a positive integer such that $IM \subseteq \mathfrak{m}^b M$. Then*

$$h_{\overline{G_I(M)}}(0) \geq b.$$

Proof. Note that

$$[H^0_{G_+}(G_I(M))]_0 = \frac{T_0}{IM},$$

where $T_0 = \cup_{n > 0}(I^{n+1}M : I^n) = I^{t+1}M : I^t \subseteq M$ for some $t \gg 0$. Set $\widetilde{M} := M/T_0$. Suppose that

$$\mathfrak{m}^{b-1}\widetilde{M} = \mathfrak{m}^b \widetilde{M}.$$

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By Nakayama's Lemma, this gives $\mathfrak{m}^{b-1}\widetilde{M} = 0$. Consequently, $\mathfrak{m}^{b-1}M \subseteq T_0$. Then $\mathfrak{m}^{b-1}I^tM \subseteq I^{t+1}M$. Since $IM \subseteq \mathfrak{m}^bM$,

$$I^{t+1}M \subseteq \mathfrak{m}^bI^tM \subseteq \mathfrak{m}I^{t+1}M \subseteq I^{t+1}M.$$

Hence $I^{t+1}M = \mathfrak{m}I^{t+1}M$. By Nakayama's Lemma, we get $I^{t+1}M = 0$. This implies $\dim(M) \leq 0$, a contradiction. Therefore $\mathfrak{m}^{b-1}\widetilde{M} \neq \mathfrak{m}^b\widetilde{M}$ and we obtain a strict chain of submodules of $\widetilde{M}$:

$$\widetilde{M} \supsetneq \mathfrak{m}\widetilde{M} \supsetneq \cdots \supsetneq \mathfrak{m}^b\widetilde{M}.$$

Since $\overline{G_I(M)}_0 = M/T_0 = \widetilde{M}$, we must have

$$h_{\overline{G_I(M)}}(0) = \ell(\widetilde{M}) \geq \ell(\widetilde{M}/\mathfrak{m}^b\widetilde{M}) \geq b.$$

$\square$

The following result was given by L. T. Hoa [8, Theorem 5.2] for rings, but its proof also holds for modules.

Lemma 1.8. *Let M be a module of positive depth. Then*

$$a_0(G_I(M)) \leq a_1(G_I(M)) - 1.$$

We are now in the position to show the main results of this section.

Proposition 1.9. *Let M be an one-dimensional Cohen-Macalay module. Let b be the maximal integer such that $IM \subseteq \mathfrak{m}^bM$. Then*

$$\mathrm{reg}(G_I(M)) \leq e_0 - b.$$

Proof. Since $\mathrm{depth}(M) > 0$, by Lemma 1.8,

$$\mathrm{reg}(G_I(M)) = \mathrm{reg}^1(G_I(M)) = \mathrm{reg}(\overline{G_I(M)}).$$

Since $\dim(M) = 1$, $\overline{G_I(M)}$ is a Cohen-Macaulay module. Assume that $x^* \in I/I^2$ is an $\overline{G_I(M)}$-regular element. Since $G_I(M)$ is generated in degree 0, $d(\overline{G_I(M)}/x^*\overline{G_I(M)}) = 0$. Applying Lemma 1.4 (i) and Lemma 1.8 we get

(5) $$\mathrm{reg}(\overline{G_I(M)}) \leq e_0 - h_{\overline{G_I(M)}}(0) \leq e_0 - b.$$

$\square$

Remark 1.10. Under the above assumption, Linh [11, Corollary 4.5 (ii)] showed that $\mathrm{reg}(G_I(M)) \leq e_0 - 1$. Hence, if $b > 1$ the above result improves Linh's bound.

The following result was formulated in the introduction.

Theorem 1.11. *Let M be an one-dimensional module. Let b be the maximal integer such that $IM \subseteq \mathfrak{m}^bM$. Then*

$$\mathrm{reg}(G_I(M)) \leq \binom{e_0 - b + 2}{2} - e_1 - 1.$$

Proof. Let $L := H^0_{\mathfrak{m}}(M)$. By [11, Lemma 4.3] (or see [4, Lemma 1.9]) we have

$$\mathrm{reg}(G_I(M)) \leq \mathrm{reg}(G_I(\overline{M})) + \ell(L).$$

By [13, Proposition 2.3], it implies that $\ell(L) = \overline{e}_1 - e_1$ and $\overline{e}_0 = e_0$, where $\overline{e}_0 := e_0(I, \overline{M})$ and $\overline{e}_1 := e_1(I, \overline{M})$. Note that $IM \subseteq \mathfrak{m}^b M$ implies $I\overline{M} \subseteq \mathfrak{m}^b \overline{M}$. Since $\overline{M}$ is a Cohen-Macaulay module, Lemma 1.5 (i) says that $\overline{e}_1 \leq \binom{e_0 - b + 1}{2}$. This gives

$$\ell(L) \leq \binom{e_0 - b + 1}{2} - e_1.$$

By Proposition 1.9, $\operatorname{reg}(G_I(\overline{M})) \leq e_0 - b$. Hence,

$$\operatorname{reg}(G_I(M)) \leq e_0 - b + \binom{e_0 - b + 1}{2} - e_1 = \binom{e_0 - b + 2}{2} - e_1 - 1.$$

$\square$

2. Extremal case

In this section, let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and I an $\mathfrak{m}$-primary ideal. Let M be a finite generated A-module. The aim of this section is to give characterizations for the case that the bound in Theorem 1.11 is attained.

From the Grothendieck-Serre formula (1) and Lemma 1.8 we immediately get

Lemma 2.1. *Let M be a module of positive depth. Then*

$$p(G_I(M)) \leq \operatorname{reg}(G_I(M)) - 1.$$

Lemma 2.2. *Let M be a module with $\dim(M) \geq 1$. Let $L := H^0_{\mathfrak{m}}(M)$ and $\overline{M} := M/L$. Then the following conditions are equivalent:*

(i) $\operatorname{reg}(G_I(M)) = \operatorname{reg}(G_I(\overline{M})) + \ell(L)$;

(ii) $HP_K(z) = \sum_{\operatorname{reg}(G_I(\overline{M}))+1}^{\operatorname{reg}(G_I(\overline{M}))+\ell(L)} z^i$, *where* $K = \oplus_{n \geq 0} \frac{I^{n+1}M + L \cap I^n M}{I^{n+1}M}$.

Proof. Note that $\ell(K) = \ell(L)$. If $L = 0$, there is nothing to prove. Assume that $L \neq 0$.

(ii) $\Longrightarrow$ (i): Since $\dim(K) = 0$, $\operatorname{reg}(K) = \operatorname{reg}(G_I(\overline{M})) + \ell(L)$. From the short exact sequence

$$(6) \qquad\qquad 0 \longrightarrow K \longrightarrow G_I(M) \longrightarrow G_I(\overline{M}) \longrightarrow 0,$$

we see that (see e. g., [6, Corollary 20.19 (d)])

$$\operatorname{reg}(G_I(M)) = \max\{\operatorname{reg}(K), \operatorname{reg}(G_I(\overline{M}))\} = \operatorname{reg}(G_I(\overline{M})) + \ell(L).$$

(i) $\Longrightarrow$ (ii): Set $a = \operatorname{reg}(G_I(\overline{M}))$ and $m = \max\{t \mid K_t \neq 0\}$. It was proved in [4, Lemma 1.9] that $K_t \neq 0$ for all $a + 1 \leq t \leq m$. By (6),

$$a + \ell(L) = \operatorname{reg}(G_I(\overline{M})) + \ell(L) = \operatorname{reg}(G_I(M)) \leq \max\{a, m\}$$
$$= a + \max\{0, m - a\} \leq a + \ell(K) = a + \ell(L).$$

This implies $\ell(K) = m - a$, and so $\ell(K_t) = 1$ for $a + 1 \leq t \leq m$ and $\ell(K_t) = 0$ for all other values.

$\square$

Now we can state and prove the main results of this section. It is interesting to mention that the equality $e_1 = \binom{e_0 - b + 1}{2}$ implies the Cohen-Macaulayness of $G_I(M)$.

Theorem 2.3. *Let M be an one-dimensional Cohen-Macalay module. Let b be a positive integer such that $IM \subseteq \mathfrak{m}^b M$. Then the following conditions are equivalent:*

(i) $\operatorname{reg}(G_I(M)) = \binom{e_0-b+2}{2} - e_1 - 1$;

(ii) $HP_{I,M}(z) = \frac{b+\sum_{i=1}^{e_0-b} z^i}{1-z}$;

(iii) $e_1 = \binom{e_0-b+1}{2}$;

(iv) $\operatorname{reg}(G_I(M)) = e_0 - b$ and $G_I(M)$ is a Cohen-Macaulay module.

Moreover, if one of the above condition holds then $b = \max\{t \mid IM \subseteq m^t M\}$.

Proof. The last statement follows from Remark 1.6 (i).

(ii) $\iff$ (iii) is the Rossi-Valla result (see Lemma 1.5 (ii)).

(i) $\implies$ (iii): Since M is a Cohen-Macaulay module, by Proposition 1.9, $\operatorname{reg}(G_I(M)) \le e_0 - b$. Hence $\binom{e_0-b+2}{2} - e_1 - 1 \le e_0 - b$, or equivalently $e_1 \ge \binom{e_0-b+1}{2}$. By Lemma 1.5 (i), we then get $e_1 = \binom{e_0-b+1}{2}$.

(ii) $\implies$ (iv): Note that we always have $e_0 \ge b$. By Remark 1.1 (ii), $p(G_I(M)) = e_0 - b - 1$. Applying Lemma 2.1, we get $e_0 - b \le \operatorname{reg}(G_I(M))$. Combining with Proposition 1.9 we can conclude that $\operatorname{reg}(G_I(M)) = e_0 - b$.

By Lemma 1.8, $\operatorname{reg}(G_I(M)) = \operatorname{reg}(\overline{G_I(M)}) = e_0 - b$. From (5) we get $h_{\overline{G_I(M)}}(0) = b$. By Lemma 1.4 (ii) this implies

$$
h_{\overline{G_I(M)}}(t) = \begin{cases} b & \text{if } t = 0, \\ t + b & \text{if } 1 \le t \le e_0 - b - 1, \\ e_0 & \text{if } t \ge e_0 - b. \end{cases}
$$

Consequently,

$$
HP_{\overline{G_I(M)}}(z) = \frac{b + \sum_{i=1}^{e_0-b} z^i}{1 - z} = HP_{I,M}(z).
$$

Hence $G_I(M) = \overline{G_I(M)}$ and $G_I(M)$ is a Cohen-Macaulay module.

(iv) $\implies$ (i): Since $G_I(M)$ is a Cohen-Macaulay module and $\operatorname{reg}(G_I(M)) = e_0 - b$, using Lemma 1.4 (ii) we get $HP_{I,M}(z) = \frac{b+\sum_{i=1}^{e_0-b} z^i}{1-z}$. By virtue of the equivalence of (ii) and (iii), it follows that $\binom{e_0-b+1}{2} = e_1$. Therefore

$$
\begin{aligned}
\operatorname{reg}(G_I(M)) = e - b &= e - b + \binom{e_0-b+1}{2} - e_1 \\
&= \binom{e_0-b+2}{2} - e_1 - 1.
\end{aligned}
$$

$\square$

The following example shows that the assumption $G_I(M)$ being a Cohen-Macalay module in (iv) of the above theorem is essential.

Example 2.4. Let $A = k[[t^3, t^4, t^5]] \cong k[[x, y, z]]/(x^3 - yz, xz - y^2, x^2 y - z^2)$, where k is a field. Let $I = (t^3, t^4)$. We have $b = 1$. Using CocoA package [3], we can compute $HP_{I,A}(z) = \frac{2+z^2}{1-z}$. Hence $e_0 = 3$, $e_1 = 2$. By Lemma 1.1, $p(G_I(A)) = 1$. Using also Lemma 2.1, we then get $\operatorname{reg}(G_I(A)) \ge 2$. By Proposition 1.9, $\operatorname{reg}(G_I(A)) \le e_0 - b = 2$, which yields $\operatorname{reg}(G_I(A)) = e_0 - b = 2$. However $2 = e_1 \ne \binom{e_0-b+1}{2} = 3$. Note that, by Theorem 2.3, $G_I(A)$ in this example cannot be a Cohen-Macalay ring.

Theorem 2.5. *Let M be an one-dimensional module and $\operatorname{depth}(M) = 0$. Let b be a positive integer such that $IM \subseteq \mathfrak{m}^b M$. Then the following conditions are equivalent:*

(i) $\mathrm{reg}(G_I(M)) = \binom{e_0-b+2}{2} - e_1 - 1$;

(ii) $HP_{I,M}(z) = \dfrac{b+\sum_{i=1}^{e_0-b+1} z^i - z^{\binom{e_0-b+2}{2}-e_1}}{1-z}$.

Moreover, if one of the above condition holds then $b = \max\{t \mid IM \subseteq m^t M\}$.

Proof. (i) $\implies$ (ii): For simplicity we set $\overline{M} := M/L$, $\overline{e}_0 := e_0(I,\overline{M}) = e_0$ and $\overline{e}_1 := e_1(I,\overline{M})$, where $L := H^0_{\mathfrak{m}}(M)$. Analyzing the proof of Theorem 1.11 we see that the condition (i) implies

(7)
$$\mathrm{reg}(G_I(\overline{M})) = e_0 - b,$$

(8)
$$\overline{e}_1 = \binom{e_0-b+1}{2},$$

and

(9)
$$\mathrm{reg}(G_I(M)) = \mathrm{reg}(G_I(\overline{M})) + \ell(L).$$

By [13, Proposition 2.3] and (8) we get

(10)
$$\ell(L) = \overline{e}_1 - e_1 = \binom{e_0-b+1}{2} - e_1.$$

Using Lemma 2.2, the equality (9) implies

(11)
$$HP_K(z) = \sum_{\mathrm{reg}(G_I(\overline{M}))+1}^{\mathrm{reg}(G_I(\overline{M}))+\ell(L)} z^i,$$

where $K = \oplus_{n\geq 0}\frac{I^{n+1}M+L\cap I^n M}{I^{n+1}M}$. Combining (7), (8) and (11) we get

$$HP_K(z) = \sum_{e_0-b+1}^{\binom{e_0-b+2}{2}-e_1-1} z^i.$$

Using Lemma 1.5(ii) and (7) we have

$$HP_{I,\overline{M}}(z) = \frac{b+\sum_{i=1}^{e_0-b} z^i}{1-z}.$$

Hence, using the short exact sequence (6) we conclude that

$$HP_{I,M}(z) = HP_{I,\overline{M}} + HP_K(z)$$

$$= \frac{b+\sum_{i=1}^{e_0-b} z^i}{1-z} + \sum_{e_0-b+1}^{\binom{e_0-b+2}{2}-e_1-1} z^i$$

$$= \frac{b+\sum_{i=1}^{e_0-b+1} z^i - z^{\binom{e_0-b+2}{2}-e_1}}{1-z}.$$

(ii) $\implies$ (i): By Lemma 1.5 (i), $\binom{e_0-b+2}{2} - e_1 > e_0 - b + 1$. Hence, by Lemma 1.1, $p(G_I(M)) = \binom{e_0-b+2}{2} - e_1 - 1$. By Remark 1.2 (i), we obtain

$$\binom{e_0-b+2}{2} - e_1 - 1 \leq \mathrm{reg}(G_I(M)).$$

Combining with Theorem 1.11 we then get $\mathrm{reg}(G_I(M)) = \binom{e_0-b+2}{2} - e_1 - 1$.

For the last statement we see that in this case the equality (8) holds. By Remark 1.6 (i), $I\overline{M} \nsubseteq \mathfrak{m}^{b+1}\overline{M}$. This implies $IM \nsubseteq \mathfrak{m}^{b+1}M$ and $b = \max\{t \mid IM \subseteq \mathfrak{m}^t M\}$. $\square$

There are many examples of one-dimension Cohen-Macaulay rings $(A, \mathfrak{m})$ and $\mathfrak{m}$-primary ideals such that $e_1 = \binom{e_0-b+1}{2}$. Hence, by Theorem 2.3, the upper bound $\binom{e_0-b+2}{2} - e_1 - 1$ is sharp. The following example show that this bound is also attained in the non-Cohen-Macaulay case.

Example 2.6. Let $A = k[[x,y]]/(x^s y^{u+v}, x^{s+1}y^u)$, where $s, u, v \in \mathbb{N}$ and $v > 0$. Then $G_{\mathfrak{m}}(A) \cong k[x,y]/(x^s y^{u+v}, x^{s+1}y^u)$ and $b = 1$. It is easy to see that

$$HP_{\mathfrak{m},A}(z) = \frac{\sum_{i=0}^{s+u} z^i - z^{s+u+v}}{1-z}, e_0 = s+u, e_1 = \frac{(s+u)(s+u-1)}{2} - v,$$

and $\operatorname{reg}(G_{\mathfrak{m}}(A)) = s+u+v-1$. These equalities show that all conditions in Theorem 2.5 hold.

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REFERENCES

[1] M. P. Brodmann and R. Y. Sharp, Local cohomology: an algebraic introduction with geometric applications. Cambridge Studies in Advanced Mathematics, 60. Cambridge University Press, Cambridge, 1998.

[2] W. Bruns and J. Herzog, Cohen-Macaulay rings, Cambridge University Press, (1993).

[3] CoCoA: a system for doing Computations in Commutative Algebra, Available at http://cocoa.dima.unige.it

[4] L. X. Dung and L. T. Hoa, *Castelnuovo-Mumford regularity of associated graded modules and fiber cones of filtered modules*, Comm. Algebra. **40** (2012), 404-422.

[5] L. X. Dung and L. T. Hoa, *Dependence of Hilbert coefficients*, preprint.

[6] D. Eisenbud, Commutative Algebra with a View toward Algebraic Geometry, Springer-Verlag, (1995).

[7] J. Elias, M. E. Rossi and G. Valla, *On the coefficients of the Hilbert-Samuel for a Cohen-Macaulay module*, J. London Math. Soc(2) **25**(1982), 449-457

[8] L. T. Hoa, *Reduction numbers of equimultiple ideals*, J. Pure Appl. Algebra, **109**(1996), 111-126.

[9] L. T. Hoa and E. Hyry, *Castelnuovo-Mumford regularity of initial ideals*. J. Symb. Comp. **38**(2004), 1327-1341.

[10] D. Kirby and H.A. Mehran, *A note on the coefficients of the Hilbert-Samuel polynomial for a Cohen-Macaulay module*, J. London Math. Soc. (2) **25** (1982), 449-457.

[11] C. H. Linh, *Upper bound for Castelnuovo-Mumford regularity of associated graded modules*, Comm. Algebra. **33** (2005), 1817-1831.

[12] M. E. Rossi, N. V. Trung and G. Valla, *Castelnuovo-Mumford regularity and extended degree*, Trans. Amer. Math. Soc. **355** (2003), 1773-1786.

[13] M. E. Rossi and G. Valla, *Hilbert function of filtered modules*, Lect. Notes UMI 9, Springer-Verlag 2010.

[14] V. Trivedi, *Hilbert functions, Castelnuovo-Mumford regularity and uniform Artin-Rees numbers*, Manuscripta Math. (4) **94** (1997), 485-499.